Termite and ant diffusions in the d-dimensional lattice trapping model

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Abstract. The coherent medium approximation of Odagaki and Lax is generalised to the trapping model. The frequency-dependent diffusion constant in the d-dimensional hypercubic lattice is studied when the jump rate obeys a bimodal distribution. The coherent medium approximation gives the correct static diffusion constant. The imaginary part of the AC part of the diffusion constant vanishes linearly in frequency ω when $d > 2$, as $\omega \ln \omega$ when $d = 2$ and as $\omega^{d/2}$ when $d < 2$. The corresponding real part vanishes quadratically in frequency when $d > 4$, as $\omega^2 \ln \omega$ when $d = 4$ and as $\omega^{d/2}$ when $d < 4$. The termite limit is studied by taking the limit that one of the two jump rates (probability p) becomes infinite. In the termite diffusion the static diffusion constant is critical at $p = 1$ and the critical exponents are the same as those for the termite diffusion in the hopping model. The ant (or ant lion) limit is defined by the limit of one jump rate being zero. The imaginary and real parts of the AC diffusion constant vanish linearly and quadratically in frequency, respectively. The critical exponent of the leading real part of the diffusion constant is one less than the corresponding exponent for the ant diffusion in the hopping model below the percolation threshold, while the leading imaginary part has the same critical exponent as those in the hopping model.

1. Introduction

Stochastic transport in disordered media has been attracting wide interest (e.g. Haus and Kehr 1987a). Carriers in the stochastic transport regime are assumed to obey a random walk equation in a random environment. Two models for the jump rates between two localised sites in a random walk equation have been studied extensively: one is the hopping model where the jump rate $w_{s',s}$ from site s to site s' satisfies the symmetry† $w_{s,s'} = w_{s',s}$ and the other is the trapping model where the jump rate $w_{s',s}$ is determined by the property of site s and does not depend on the property of site s' provided the jump from s to s' is possible. The trapping model can be considered as a hopping model with correlated asymmetric jump rates.

Continuous models have also been utilised in the analysis of stochastic transport. de Gennes (1980) studied a diffusion process in a random mixture of 'normal metals' and 'superconductors', which he called a termite problem. His result does not show percolation phenomena. de Gennes' work has been followed by that of Coniglio and Stanley (1984), Adler *et al* (1985) and Bunde *et al* (1985). A close comparison between these models in one dimension has also been carried out (Leyvraz *et al* 1986). Odagaki (1986) analysed the dynamic diffusion of a termite problem using the hopping random

† More precisely, the detailed balance must be obeyed.

walk model. His result shows a percolation phenomenon in agreement with previous studies except for de Gennes' treatment.

In this paper, I study a diffusion process in the trapping random walk model where the jump rate obeys a bimodal distribution, using a generalised coherent medium approximation (Odagaki and Lax 1981). The limit that one type of jump rate is infinity is considered to be the termite diffusion in the trapping model. The static diffusion constant is critical at the limiting value of the probability ($p = 1$). This coincides with the result suggested by de Gennes (1980). The AC parts of the diffusion constant are shown to be critical at the same place as is the DC part. I also analyse the AC diffusion constant for the percolation limit in which one type of jump rate vanishes, i.e. a perfect trap. In a sense, this process can be viewed as ant diffusion with 'ant lions' whose larvae make a trap in the ground to catch ants.

In §2 I develop a generalised coherent medium approximation suitable for the trapping model. This approximation turns out to be identical to the one proposed recently by Haus and Kehr (1987b). The system with a bimodal distribution is analysed in §3. The termite limit and the ant lion limit are studied in §§ 4 and 5, respectively, and § 6 contains discussion.

2. Coherent medium approximation

I consider the trapping model in the d-dimensional hypercubic lattice where a carrier can jump only between nearest neighbours. The probability $P(s, t|s_0, 0)$ that a carrier which started from site s_0 at time $t = 0$ is at site s at time t is assumed to obey the time development determined by

$$\frac{\partial P(s, t|s_0, 0)}{\partial t} = \sum_{s'}' \{w_{s'} P(s', t|s_0, 0) - w_s P(s, t|s_0, 0)\}. \tag{1}$$

The summation is taken over the nearest neighbours of site s and the jump rate w_s leaving from site s is a random quantity. The Laplace transform of $P(s, t|s_0, 0)$

$$\tilde{P}(s, u|s_0) = \int_0^\infty P(s, t|s_0, 0)e^{-ut}\, dt \tag{2}$$

then obeys

$$(u + zw_s)\tilde{P}(s, u|s_0) - \sum_{s'}' w_{s'}\tilde{P}(s, u|s_0) = \delta_{s,s_0} \tag{3}$$

where δ_{s,s_0} is the Kronecker δ function and z is the coordination number of the lattice. It is convenient to introduce a 'random walk Hamiltonian'

$$H = -\sum_s |s\rangle zw_s\langle s| + \sum_{s' \neq s}' |s\rangle w_s\langle s'| \tag{4}$$

with an orthonormal basis set $\{|s\rangle\}$. The solution to (3) is formally given by

$$\tilde{P}(s, u|s_0) = \langle s|(u - H)^{-1}|s_0\rangle. \tag{5}$$

Following the philosophy of the coherent medium approximation (Odagaki and Lax 1981), I introduce an approximate system H_A

$$H_A = H_C + V \tag{6}$$

where H_C is a coherent medium with a u-dependent coherent jump rate w_C everywhere

$$H_C = -\sum_s |s\rangle z w_C \langle s| + \sum_{s'\neq s}' |s\rangle w_C \langle s'| \tag{7}$$

and

$$V = \sum_{\substack{s \\ \text{NN of } s_0}} |s\rangle (w_{s_0} - w_C)\langle s_0| + |s_0\rangle z(w_C - w_{s_0})\langle s_0|. \tag{8}$$

Namely, in H_A all the jump rates except for those leaving site s_0 are the coherent one which is to be determined self-consistently by requiring

$$\langle (u - H_A)^{-1}\rangle = (u - H_C)^{-1} \tag{9}$$

where the angular bracket $\langle\rangle$ denotes the ensemble average over the distribution of the jump rate $w_0 \equiv w_{s_0}$. Since the 'scattering potential' V contains only $z+1$ bases (s_0 and its z neighbours), it is easy to show that the condition (9) is reducible to a $(z+1) \times (z+1)$ matrix equation

$$\langle \tilde{V}(\tilde{1} - \tilde{P}_C \tilde{V})^{-1}\rangle = 0 \tag{10}$$

where $\tilde{1}$ is the $(z+1) \times (z+1)$ unit matrix, $\tilde{V}_{00} = z(w_C - w_0)$, $\tilde{V}_{j0} = w_0 - w_C$ $(j = 1, \cdots, z)$ and the rest of the elements of $\tilde{V}$ are zero, and $(\tilde{P}_C)_{i,j} \equiv \tilde{P}_{ij} = \langle s_i|(u - H_C)^{-1}|s_j\rangle$. Using the identity

$$(u + z w_C)\bar{P}_{00} - w_C \sum_{j=1}^{z} \bar{P}_{0j} = 1 \tag{11}$$

one can easily show that the $(z+1)^2$ conditions in equation (10) reduce to a single equation

$$\left\langle \frac{w_0 - w_C}{w_C + (w_0 - w_C)(1 - u\bar{P}_{00})}\right\rangle = 0. \tag{12}$$

This is the same equation derived by Haus and Kehr (1987). The diagonal element $\bar{P}_{00}$ is related to the Hilbert transformation of the density of states $n(x)$ by

$$\bar{P}_{00} = F(1 + u/z w_C)/z w_C \tag{13}$$

and

$$F(\xi) = \int_{-\infty}^{\infty} \frac{n(x)}{\xi - x}\,\mathrm{d}x. \tag{14}$$

Here, $n(x)$ is the density of states for the regular lattice

$$n(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} [J_0(y)]^d\, \mathrm{e}^{\mathrm{i}xy}\,\mathrm{d}y. \tag{15}$$

$J_0(y)$ being the zeroth-order Bessel function of the first kind.

In the coherent medium approximation the diffusion constant is simply given by

$$D(\omega) = a^2 w_C(\mathrm{i}\omega) \tag{16}$$

where a is the lattice constant.

3. Bimodal distribution

First, I consider the bimodal distribution for w_s

$$P(w_s) = p\delta(w_s - w_p) + q\delta(w_s - w_q) \tag{17}$$

where $p + q = 1$, $0 \leq p \leq 1$. For this distribution of w_0, (12) becomes

$$u\bar{P}_{00} = \frac{w_p w_q - w_C(qw_p + pw_q)}{(w_C - w_p)(w_C - w_q)} . \tag{18}$$

I am primarily interested in the frequency-dependent diffusion constant near the static limit. It is known that $F(1+\varepsilon)$ shows the following properties when $\varepsilon \simeq 0$ (Odagaki *et al* 1983)

(i)	$d < 2$	$\Gamma(d/2)\Gamma(1 - d/2)h(1)\varepsilon^{d/2-1} + \dots$
(ii)	$d = 2$	$-h(1)\ln\varepsilon + \dots$
(iii)	$2 < d < 4$	$m_1 - \Gamma(d/2 - 1)\Gamma(2 - d/2)h(1)\varepsilon^{d/2-1} + \dots$
(iv)	$d = 4$	$m_1 + h(1)\varepsilon\ln\varepsilon + \dots$
(v)	$4 < d$	$m_1 - m_2\varepsilon + \dots$

$$\tag{19}$$

where $\Gamma(x)$ is the gamma function,

$$h(1) = \lim_{x \to 1} \frac{n(x)}{(1-x)^{d/2-1}} \tag{20}$$

and

$$m_k = \int_{-\infty}^{x} \frac{n(x)}{(1-x)^k}\,\mathrm{d}x. \tag{21}$$

It is straightforward to show that the static diffusion constant is given by

$$D(0)/a^2 = \frac{w_p w_q}{qw_p + pw_q} = \langle w_s^{-1}\rangle^{-1} \tag{22}$$

for any dimensions. This result agrees with the exact result (Haus *et al* 1982). Figure 1 shows the p dependence of the static diffusion constant for various values of w_p and w_q. The AC part of the diffusion constant can also be determined by a careful analysis of (18). Table 1 summarises the results where the AC part near the static limit is written as

$$[D(\omega) - D(0)]/a^2 = A(p)f(\omega)\mathrm{i} + B(p)g(\omega). \tag{23}$$

Note that when $w_p = w_q$ then $A(p)$ and $B(p)$ vanish as expected.

4. The termite limit

The termite limit is defined as the limit of $w_p = \infty$ with $w_q = 1$ (constant). The static diffusion constant in this limit is given by

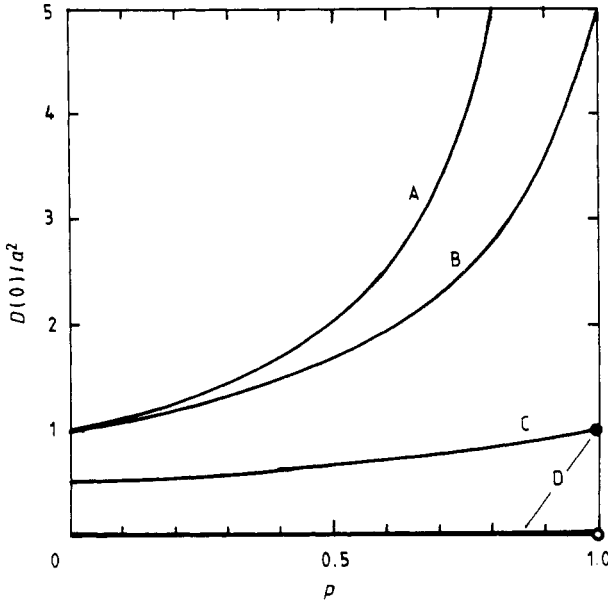$$D(0)/a^2 w_q = (1-p)^{-1} \tag{24}$$

Figure 1. The static diffusion constant $D(0)/a^2$ plotted against p for the system with the bimodal distribution for the jump rate in the trapping model for various values of w_p and w_q. (A) $w_p = \infty$, $w_q = 1$ corresponding to the termite limit; (B) $w_p = 5$, $w_q = 1$; (C) $w_p = 1$, $w_q = 0.5$; (D) $w_p = 1$, $w_q = 0$ corresponding to the ant lion limit.

in agreement with the exact result (Haus *et al* 1982). This relation is also shown in figure 1. I define critical exponents s, v and ϕ by ($p_c = 1$) (Straley 1976):

$$D(0) \sim (p_c - p)^{-s} \qquad\qquad \text{when } p \to p_c - 0^+$$

$$D(0) \sim w_p (w_q/w_p)^{v} \qquad\qquad \text{when } p = p_c \qquad\qquad (25)$$

$$D(0) \sim w_p (w_q/w_p)^{v} T_- \left[\left(\frac{p_c - p}{w_q} \right) \left(\frac{w_p}{w_q} \right)^{\phi} \right] \qquad \text{when } w_p \to \infty, \ p \to p_c - 0^+ .$$

One can easily find $s = 1$, $v = 0$, $\phi = 1$ and $T_-(x) = 1/x$ which satisfy the scaling relation $s\phi + v = 1$, the same relation proposed by Bunde *et al* (1985) for their termite model.

The AC part of the generalised diffusion constant in the termite limit can also be obtained from table 1. The frequency dependence of the imaginary and real parts remain the same as given in table 1. The coefficients $A_t(p)$ and $B_t(p)$ of the leading imaginary and real parts near the static limit of $D(\omega)$ show critical behaviour. I define critical exponents $\mu_t(d)$ and $\nu_t(d)$ by

$$A_t(p) \sim (p_c - p)^{-\mu_t(d)} \qquad\qquad\qquad\qquad (26)$$

$$B_t(p) \sim (p_c - p)^{-\nu_t(d)} \qquad\qquad\qquad\qquad (27)$$

with $p_c = 1$. I found $\mu_t(d) = 1$ for $d \geq 2$ and $\mu_t(d) = 2 - d/2$ for $d \leq 2$, and $\nu_t(d) = 0$ for $d \geq 4$ and $\nu_t(d) = 2 - d/2$ for $d \leq 4$. These exponents are the same as those for the termite diffusion in the hopping model (Odagaki 1986).

Table 1. The AC parts of the low-frequency diffusion constant $[D(\omega) - D(0)]/a^2 = A(p)f(\omega)\mathrm{i} + B(p)g(\omega)$ determined by the present approximation for the bimodal distribution. Note that $A(p)$ and $B(p)$ vanish when $w_p = w_q$.

d	$A(p)$	$f(\omega)$	$B(p)$	$g(\omega)$
$d > 4$		ω	$\dfrac{pq(w_p - w_q)^2}{z^2 w_p w_q (qw_p + pw_q)^3}[m_1^2(q^2 w_p - p^2 w_q)(w_p - w_q) - m_2(qw_p + pw_q)]$	ω^2
$d = 4$	$\dfrac{m_1 pq(w_p - w_q)^2}{(qw_p + pw_q)^2}$	ω	$\dfrac{h(1)pq(w_p - w_q)^2}{z^2 w_p w_q (qw_p + pw_q)}$	$\omega^3 \ln \omega$
$4 > d > 2$	$-\dfrac{h(1)pq(w_p - w_q)^2}{z(qw_p + pw_q)^2}$		$\dfrac{\Gamma(d/2 - 1)\Gamma(2 - d/2)h(1)pq(w_p - w_q)^2}{z^{d/2}(w_p w_q)^{d/2 - 1}(qw_p + pw_q)^{3 - d/2}}$	$\omega^{d/2}$
$d = 2$		$\omega \ln \omega$	$-\dfrac{\pi h(1)pq(w_p - w_q)^2}{2z(qw_p + pw_q)^2}$	
$2 > d \geqslant 1$	$\dfrac{\Gamma(d/2)\Gamma(1 - d/2)h(1)pq(w_p - w_q)^2}{z^{d/2}(w_p w_q)^{d/2 - 1}(qw_p + pw_q)^{3 - d/2}}\cos\left(\dfrac{\pi d}{4}\right)$	$\omega^{d/2}$	$\dfrac{\Gamma(d/2)\Gamma(1 - d/2)pq(w_p - w_q)^2}{z^{d/2}(w_p w_q)^{d/2 - 1}(qw_p + pw_q)^{3 - d/2}}\sin\left(\dfrac{\pi d}{4}\right)$	$\omega^{d/2}$

5. The ant lion limit

I define the ant lion limit by $w_q = 0$ setting $w_p = 1$. Obviously $D(0) = 0$ except when $p = 1$. If one defines a critical exponent t by $D(0) \sim w_p(p - p_c)^t (p \geq p_c)$ as usually done in the problem of random register networks, this result implies $t = 0$ and $p_c = 1$. Furthermore, one can write $D(0)$ at $p = p_c$ as $D(0) \sim w_p(w_q/w_p)^v$ with $v = 0$ and $D(0)$ in the limit of $w_q \to 0$ and $p \to p_c - 0^+$ as

$$D(0) \sim w_p(w_q/w_p)^v A_- \left[\left(\frac{p_c - p}{p_c} \right) \left(\frac{w_p}{w_q} \right)^\phi \right]$$

with $v = 0$ and $\phi = 1$ (Straley 1976). Thus, the critical exponents t, v and ϕ satisfy a scaling relation $t\phi = v$. Note that this and the relation in the previous section imply $v = t/(s + t)$ and $\phi = 1/(s + t)$ (Bunde *et al* 1985).

The AC part of the diffusion constant behaves as

$$D(\omega)/a^2 \sim A_a(p)\mathrm{i}\omega + B_a(p)\omega^2 \tag{28}$$

where $A_a(p)$ is the solution of the equation

$$zA_a(p)(1-p) = F\left(1 + \frac{1}{zA_a(p)} \right) \tag{29}$$

and $B_a(p)$ is given by

$$B_a(p) = \frac{z^2(1-p)(A_a(p))^4}{z^2(1-p)(A_a(p))^2 - F'(1 + 1/zA_a(p))} \tag{30}$$

with $F'(\xi) \equiv \mathrm{d}F(\xi)/\mathrm{d}\xi$. When the probability p is close to unity the coefficients $A_a(p)$ and $B_a(p)$ show critical behaviours

$$A_a(p) \sim (p_c - p)^{-\mu_a(d)} \tag{31}$$

$$B_a(p) \sim (p_c - p)^{-\nu_a(d)} \tag{32}$$

except when $d = 2$ for $A_a(p)$ and $d = 4$ for $B_a(p)$ where a logarithmic behaviour appears. I found $\mu_a(d) = 2/d$ for $d < 2$, $A_a(p) \sim (p_c - p)^{-1} \ln(p_c - p)$ for $d = 2$ and $\mu_a(d) = 1$ for $d > 2$. Similarly, $\nu_a(d) = 4/d$ for $d < 2$, $B_a(p) \sim (p_c - p)^{-2}[\ln(p_c - p)]^2$ and $\nu_a(d) = 2$ for $d > 2$. These critical exponents can be compared to those for the hopping conduction in the d-dimensional lattice bond percolation process below p_c (Odagaki *et al* 1983). The exponents $\mu_a(d)$ are the same as those for the percolation model and $\nu_a(d)$ is one less than the corresponding critical exponent. This is due to the fact that $p_c = 1$.

6. Discussion

In this paper I have studied the trapping model using the generalised coherent medium approximation which turns out to be identical to the approximation proposed recently by Haus and Kehr (1987b). As an example, the bimodal distribution, in particular the termite limit and the ant lion limit, was investigated. The static diffusion constant in the present approximation agrees with the exact result, suggesting excellence of the present approximation. The static diffusion constant in the termite limit is critical at $p = 1$ and not at the percolation point. This result coincides with the suggestion made

by de Gennes (1980) for the so-called 'Paris termite'. This is conceivable since in the termite limit a carrier jumps out immediately from 'superconducting' regions and performs a random walk in 'normal' regions, which was in fact the assumption used by de Gennes. Thus the static diffusion constant is always finite if any 'normal' region exists. This is very similar to the fact that $D(0) = 0$ in the ant lion limit except for $p = 1$ (this is also the case even when the other jump rate becomes infinity as in the termite model). No matter how small the fraction of the traps is, any trap, if it exists, absorbs carriers eventually and no further diffusion is possible.

The low-frequency behaviours obtained here for the termite model are the same as those obtained for the termite limit in the hopping model including the logarithmic frequency dependence at $d = 2$ and $d = 4$. The coefficients $A_i(p)$ and $B_i(p)$ show critical behavior at $p = 1$ and the associated critical exponents are the same as those for the hopping model.

The ant lion limit in the trapping model was also studied. This diffusion process can be compared to ants walking in a field with ant lions. Once an ant is trapped by an ant lion, it can never come back to the field. Thus the static diffusion constant must vanish if any number of ant lions exist in the field. The behaviour of the AC parts of the diffusion constant is also very similar to those for the ant diffusion in the hopping model below the percolation threshold. The leading real and imaginary parts of $D(\omega)$ near the static limit show critical behaviour at $p = 1$. The critical exponent for the real part is one less than the corresponding critical exponents in the hopping model. This is due to the fact that the threshold in the trapping model is $p = 1$.

Finally, there has been a study of the so-called 'Tel Aviv termite' (Adler *et al* 1985) in which the first step of the diffuser is treated differently from the subsequent walk. In the present treatment all steps are treated equivalently. Thus the present model cannot describe the Tel Aviv termite.

Acknowledgments

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